ON ROULETTE WHICH ALLOWS STAKES ON INFINITELY MANY HOLES

BY

DAVID GILAT† AND ERNST-AUGUST WEISS, JR.‡

ABSTRACT

It is shown that if a gamble γ stakes positive amounts on infinitely many holes of a subfair roulette-table, then for every $\varepsilon > 0$, there is a gamble γ^* with positive stakes on only a finite number of holes, such that $\gamma Q \leqq \gamma^* Q + \varepsilon$ for every nondecreasing function Q bounded above by 1 on $[0, \infty)$. It is deduced from this proposition that a gambler who wishes to maximize his chances to increase his current fortune by a specified amount, has no advantage in ever placing positive stakes on more than a finite number of holes on any single spin. This result settles a question left open in [1].

A *roulette-table*, as in [1], is determined by a family of pairs of numbers (w_α, r_α) with $0 < w_\alpha, r_\alpha < 1$, where α ranges over some non-empty index-set, conveniently identified with the set of *holes* on the roulette. In contrast to the roulette-table $\hat{\Gamma}$ of [1], which allows only gambles that stake positive amounts on a finite number of holes, the roulette-table, $\tilde{\Gamma}$, considered in this note, makes available at f, all gambles γ, for which the new fortune, f_β, obtained from f by using γ, is nonnegative for every hole β, and for which $W(\gamma) \equiv \Sigma_\gamma w_\alpha \leqq 1$, where, as in [1], Σ_γ means summation over the set of holes on which γ stakes a positive amount. (Note that $W(\gamma) \leqq 1$ entails that γ stakes positive amounts on at most a countable infinity of holes; observe as well that each such γ is available in $\tilde{\Gamma}$ at some fortune f.) $\hat{\Gamma}$ can now be viewed as the restriction of $\tilde{\Gamma}$ to gambles with positive stakes on only a finite number of holes. As in [1], let Γ be the further restriction of $\tilde{\Gamma}$ to gambles with a positive stake on at most one hole. Let $\tilde{U}$, $\hat{U}$, and U be the respective casino functions of $\tilde{\Gamma}$, $\hat{\Gamma}$ and Γ. Since $\Gamma \subset \hat{\Gamma} \subset \tilde{\Gamma}$, plainly $U \leqq \hat{U} \leqq \tilde{U}$.

Dubins [1] has proved

† This author was on leave from Tel Aviv University when this note was prepared. He wishes to acknowledge the hospitality of the School of Statistics at the University of Minnesota.

‡ This author is thankful to the Deutsche Forschungsgemeinschaft for its financial support.

Received December 3, 1975

$$(1) \qquad \hat{U} = U,$$

and has left unanswered the question as to whether also $\tilde{U} = U$. It is the purpose of this note to settle this question in the affirmative. We do so by proving:

THEOREM. $\tilde{U} = \hat{U}$.

Thus a gambler who wishes to maximize his changes of getting richer by a specified amount through playing at the roulette-table $\tilde{\Gamma}$, *may as well restrict himself to betting on a finite number of holes — hence, by* (1), *on only one hole — on any particular spin of the roulette.*

For the rest of this note assume

$$(2) \qquad w_\alpha \leqq r_\alpha \quad \text{for every hole} \quad \alpha,$$

for otherwise Γ, *a fortiori* $\hat{\Gamma}$ and $\tilde{\Gamma}$, is superfair, so that U, *a fortiori* $\hat{U}$ and $\tilde{U}$, is the indicator-function of the positive half line $(0,\infty)$, as is established in the general theory of superfair casinos developed in [3, ch. 4].

The proof of the Theorem in the subfair case (2), though admittedly much simpler, follows the same logical pattern as that of [1, th. 1]. A gamble γ that stakes positive amounts on a finite number of holes is *of finite order.*

PROPOSITION. *Let* $0 \leqq f \leqq 1$, *let* γ *be available in* $\tilde{\Gamma}$ *at* f. *Then for every* $\varepsilon > 0$, *there is a* γ^* *of finite order available at* f *such that*

$$(3) \qquad \gamma Q \leqq \gamma^* Q + \varepsilon$$

for all nondecreasing functions Q *bounded above by* 1 *on* $[0,\infty)$.

That the Theorem follows from the Proposition is easily seen, much in the same manner as theorem 1 in [1] is argued from Lemma 1 there. For putting, as one may, $\hat{U}$ for Q in (3), then applying the fact that $\hat{U}$ is excessive for $\hat{\Gamma}$ — as, by [3, th. 2.14.1], is the U of every house for that house — one gets

$$(4) \qquad \gamma\hat{U} \leqq \gamma^*\hat{U} + \varepsilon \leqq \hat{U}(f) + \varepsilon,$$

and thus

$$(5) \qquad \gamma\hat{U} \leqq \hat{U}(f) + \varepsilon,$$

for every f and every γ available in $\tilde{\Gamma}$ at f. Since ε in (5) is arbitrary, it means that $\hat{U}$ is excessive for $\tilde{\Gamma}$. That $\tilde{U} = \hat{U}$ now follows from the fundamental theorem [3, th. 2.12.1].

Turn now to the proof of the proposition. As in [1], let γ be given by the system of stakes $\{s_\alpha, r_\alpha\}$. Assume first the mildly restrictive condition

$$f_0 = f - \Sigma_\gamma s_\alpha r_\alpha \geqq 0\,, \tag{6}$$

which is always satisfied for γ with $W(\gamma) < 1$, because otherwise γ would not be available at f. The set $\{\alpha : s_\alpha > 0\}$, that is, the set of holes on which γ places positive stakes, may conveniently be called *the set of holes used by* γ. Since $W(\gamma) \leqq 1$, the set of holes used by γ can be partitioned into a *finite* set A and a residual (possibly empty, if γ is of finite order to begin with, in which case take $\gamma^* = \gamma$) set B with

$$\Sigma_{\alpha\in B} w_\alpha \lesssim \varepsilon\,. \tag{7}$$

Define γ^* by the system of stakes $\{s^*_\alpha r_\alpha\}$, where $s^*_\alpha = 0$ for holes α for which $s_\alpha = 0$ and also for holes α in B; on each of the (finite number of) holes α in A, let $s^*_\alpha = s_\alpha$. Plainly, γ^* is of finite order and is available at f. Since $s_\alpha \geqq s^*_\alpha$ for every hole α,

$$f_0 \leqq f^*_0\,, \tag{8}$$

where f^*_0 is for γ^*, like f_0 for γ, the value of the new fortune reached from f, when a hole α with $s^*_\alpha = 0$ comes up. Also, since the set of holes used by γ^* is a subset of those used by γ,

$$1 - W(\gamma) \leqq 1 - W(\gamma^*)\,. \tag{9}$$

Moreover, for each hole β in A,

$$f_\beta \leqq f^*_\beta\,, \tag{10}$$

because as in [1], $f_\beta = f_0 + s_\beta$ and $f^*_\beta = f^*_0 + s^*_\beta$, while $f_0 \leqq f^*_0$ by (8), whereas for holes β in A, $s^*_\beta = s_\beta$ by definition of γ^*. It is only for holes in B that f_β may exceed f^*_β, but since by (7) the *total size* of these holes is less than ε, their contribution to γQ, Q being bounded above by 1, cannot exceed ε. It is now clear that, Q being nondecreasing, (7), (8), (9) and (10) yield the desired result (3). To conclude the proof of the Proposition, only the special case of γ for which (6) is violated, remains to be settled. In this case, we first replace γ by an available γ' which satisfies (6) and *dominates* γ, then proceed to obtain γ^* from γ' as before. So suppose that γ is such that $f_0 < 0$, then $W(\gamma) = 1$, because otherwise γ would not be available. The subfairness condition (2) therefore implies

$$\Sigma_\gamma r_\alpha \geqq 1\,. \tag{11}$$

Since the availability of γ at f forces $f_\beta = s_\beta + f_0$ to be nonnegative for every hole β used by γ, we may define γ' to be given by the system of stakes $\{s'_\alpha r_\alpha\}$, where $s'_\alpha = s_\alpha + f_0$ for α with $s_\alpha > 0$, and $s'_\alpha = 0$ otherwise. A straightforward computation, using (11), then shows that $f'_0 \geqq 0$ and $f'_\beta \geqq f_\beta$ for every hole β. Thus γ' so defined is an available gamble at f which dominates γ. This establishes the Proposition and thereby completes the proof of the Theorem.

REFERENCES

1. L. E. Dubins, *On roulette when the holes are of various sizes*, Israel J. Math. **11** (1972), 153–158.

2. L. E. Dubins, *A simpler proof of Smith's Roulette Theorem*, Ann. Math. Statist. **39** (1967), 390–393.

3. L. E. Dubins and L. J. Savage, *How to Gamble if You Must*, McGraw-Hill, New York, 1965.

4. G. Smith, *Optimal strategy at roulette*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, **8** (1968), 91–100.

DEPARTMENT OF THEORETICAL STATISTICS
UNIVERSITY OF MINNESOTA
MINNEAPOLIS, MINNESOTA 55455, USA

AND

MATHEMATISCHES INSTITUT DER UNIVERSITÄT ERLANGEN-NÜRNBERG
D 852 ERLANGEN BISMARCKSTR. 1 1/2
FEDERAL REPUBLIC OF GERMANY